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THE SINE-COSINE METHODS FOR SOLVING A CLASS OF NONLINEAR FOURTH ORDER VARIANT OF A GENERALIZED CAMASSA-HOLM EQUATION

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Abstract

In this paper, we study a class of nonlinear fourth order analogue of a generalized Camassa-Holm equation by using sine-cosine method. It is shown that this class gives compactons and solitary patterns solutions.

1. Introduction

Recently, Clarkson and Priestley [1] studied a class of nonlinear fourth order partial differential equations

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$$u_{tt} = (au + bu^2)_{xx} + \gamma u u_{xxxx} + \mu u_{xxtt} + \alpha u_x u_{xxx} + \beta u_{xx}^2, \tag{1.1}$$

where α, β, γ, μ, a, b are arbitrary constants. This equation was thought of [1] as a fourth order analogue of a generalization of the Camassa-Holm equation. Further, Equation (1.1) was also considered as a Boussinesq-type equation. In [1], it was shown that (1.1) admits both conventional solitons and compactons.

Motivated by the rich treasure of the Camassa-Holm-type equation in the literature, we will study nonlinear dispersive variants $CH(n, (n, m))$ of the generalized Camassa-Holm equation of the form, respectively,

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + k_0(u^n + u^m)_{xxtt}, \tag{1.2}$$

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + (a_1 u + b_1 u^n)_{yy}$$
$$+ (k_0(u^n + u^m)_{xx} + k_1(u^n + u^m)_{yy})_{tt} \tag{1.3}$$

and

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + (a_1 u + b_1 u^n)_{yy} + (a_2 u + b_2 u^n)_{zz}$$
$$+ (k_0(u^n + u^m)_{xx} + k_1(u^n + u^m)_{yy} + k_2(u^n + u^m)_{zz})_{tt}, \tag{1.4}$$

in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions, respectively. The $CH(-n, (-n, -m))$ for all previous forms will be examined as well. Specially, when $n = m$, (1.2), (1.3) and (1.4) are called nonlinear dispersive variants $CH(n, n)$ of the generalized Camassa-Holm equation in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions, respectively, [4]. Most recently, Wazwaz [4] studied the nonlinear dispersive variants $CH(n, n)$ of the generalized Camassa-Holm equation in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions, respectively, by using sine-cosine method, it is shown that this class gives compactons, conventional solitons, solitary patterns and periodic solutions.

The sine-cosine method is one of most direct and effective algebraic method for finding exact solutions of nonlinear diffusion equations and Wazwaz developed the method of sine-cosine [2-4]. The sine-cosine

algorithm, that provides a systematic framework for many nonlinear dispersive equations, will be employed to back up our analysis to determine compactons and solitary patterns travelling waves solutions. As will be shown, the change in the physical structures of the solutions depends on whether the exponent n is positive or negative, and on the coefficients of $(u^n)''$.

In what follows, we highlight the main steps of the sine-cosine algorithm.

2. The Sine-Cosine Method

The sine-cosine algorithm admits the use of the ansätz

$$u(x, t) = \lambda \cos^{\beta}(\mu\xi), \quad |\xi| \leq \frac{\pi}{2\mu}, \tag{2.1}$$

or the ansätz

$$u(x, t) = \lambda \sin^{\beta}(\mu\xi), \quad |\xi| \leq \frac{\pi}{\mu}, \tag{2.2}$$

where λ, μ and β are parameters that will be determined. We therefore use

$$\begin{aligned} u(\xi) &= \lambda \cos^{\beta}(\mu\xi), \\ u^n(\xi) &= \lambda \cos^{n\beta}(\mu\xi), \\ (u^n)'' &= -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta - 1)\cos^{n\beta-2}(\mu\xi), \end{aligned} \tag{2.3}$$

and for (2.2) we use

$$\begin{aligned} u(\xi) &= \lambda \sin^{\beta}(\mu\xi), \\ u^n(\xi) &= \lambda \sin^{n\beta}(\mu\xi), \\ (u^n)'' &= -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta - 1)\sin^{n\beta-2}(\mu\xi). \end{aligned} \tag{2.4}$$

Substituting (2.3) or (2.4) into the integrated ODE gives a trigonometric equation of $\cos^{\beta}(\mu\xi)$ or $\sin^{\beta}(\mu\xi)$ terms. The parameters b, k, and l, are

then obtained by equating the exponents of each pair of cosine or sine, and by collecting all coefficients of the same power in $\cos^{\beta}(\mu\xi)$ or $\sin^{\beta}(\mu\xi)$, and set it equal to zero.

3. Using the Sine-Cosine Method

3.1. For positive exponents

For the $CH(n, (n, m))$ of the Camassa-Holm equation given by

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + k_0(u^n + u^m)_{xxtt}, \tag{3.1}$$

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + (a_1 u + b_1 u^n)_{yy} + (k_0(u^n + u^m)_{xx} + k_1(u^n + u^m)_{yy})_{tt}, \tag{3.2}$$

and

$$u_{tt} = (a_0 u + b_0 u^n)_{xx} + (a_1 u + b_1 u^n)_{yy} + (a_2 u + b_2 u^n)_{zz} + (k_0(u^n + u^m)_{xx} + k_1(u^n + u^m)_{yy} + k_2(u^n + u^m)_{zz})_{tt}, \tag{3.3}$$

where $a_0, a_1, a_2, k_0, k_1, k_2 > 0$, $n \geq 2$, $n \neq m > 1$. Using the wave variable $\xi = x - ct$, $\xi = x + y - ct$ and $\xi = x + y + z - ct$ carries (3.1), (3.2) and (3.3) into the ODE, respectively

$$c^2 u'' = (a_j^* u + b_j^* u^n)'' + c^2 k_j^* (u^n + u^m)^{(4)}, \tag{3.4}$$

where

$$a_j^* = \sum_{i=0}^{i=j} a_i, \quad b_j^* = \sum_{i=0}^{i=j} b_i, \quad k_j^* = \sum_{i=0}^{i=j} k_i, \quad j = 0, 1, 2. \tag{3.5}$$

Integrating (3.4) twice, using the constants of integration to be zero we find

$$(a_j^* - c^2)u + b_j^* u^n + c^2 k_j^* (u^n + u^m)'' = 0. \tag{3.6}$$

Substituting (2.1) into (3.6) gives

$$(a_j^* - c^2)\lambda \cos^{\beta}(\mu\xi) + b_j^*\lambda^n \cos^{n\beta}(\mu\xi) + c^2 k_j^*(- n^2\mu^2\lambda^n\beta^2 \cos^{n\beta}(\mu\xi)$$

$$+ n\mu^2\lambda^n\beta(n\beta - 1)\cos^{n\beta-2}(\mu\xi) - m^2\mu^2\lambda^m\beta^2 \cos^{m\beta}(\mu\xi)$$

$$+ m\mu^2\lambda^m\beta(m\beta - 1)\cos^{m\beta-2}(\mu\xi)) = 0. \tag{3.7}$$

Equation (3.7) is satisfied only if the following system of algebraic equations holds:

$$n\beta \neq 1,\ m\beta \neq 1,\ \beta = m\beta - 2,\ m\beta = n\beta - 2,\ (a_j^* - c^2)\lambda + m\mu^2\lambda^m\beta(m\beta - 1) = 0,$$

$$b_j^*\lambda^n - c^2 k_j^* n^2\mu^2\lambda^n\beta^2 = 0,\ c^2 k_j^* n\mu^2\lambda^n\beta(n\beta - 1) - c^2 k_j^* m^2\mu^2\lambda^m\beta^2 = 0, \tag{3.8}$$

or

$$n\beta \neq 1,\ m\beta = 1,\ m\beta = n\beta - 2,\ a_j^* - c^2 = 0,\ \lambda + m\mu^2\lambda^m\beta(m\beta - 1) = 0,$$

$$b_j^*\lambda^n - c^2 k_j^* n^2\mu^2\lambda^n\beta^2 = 0,\ c^2 k_j^* n\mu^2\lambda^n\beta(n\beta - 1) - c^2 k_j^* m^2\mu^2\lambda^m\beta^2 = 0. \tag{3.9}$$

Solving the system (3.8) and (3.9) give, respectively,

$$\beta = \frac{4}{n-1},\ n + 1 = 2m,\ \mu = \frac{n-1}{4cn}\sqrt{\frac{b_j^*}{k_j^*}},\ \lambda = \left(\frac{(n+1)^2}{n(3n+1)}\right)^{\frac{2}{n-1}},$$

$$c = \pm\left(\frac{a_j^* \pm \sqrt{a_0^2 + \dfrac{3(n+1)^4 b_0}{2k_0 n^3(3n+1)}}}{}\right)^{\frac{1}{2}},\ a_j^*,\ k_j^* > 0, \tag{3.10}$$

and

$$c = \pm\sqrt{a_j^*},\ \beta = \frac{1}{m},\ n = 3m,\ \mu = \frac{1}{3}\sqrt{\frac{b_j^*}{a_j^* k_j^*}},\ \lambda = \pm\left(\frac{1}{6}\right)^{\frac{1}{2m}},\ a_j^*,\ k_j^* > 0. \tag{3.11}$$

The results (3.10) and (3.11) can be easily obtained, if we also use the sine method (2.2). Combining (3.10) with (2.1) and (2.2), the following compactons solutions

$$u(x_j^*, t) = \left\{ \frac{(n+1)^2}{n(3n+1)} \cos^2 \left[\frac{n-1}{4cn} \sqrt{\frac{b_j^*}{k_j^*}} (x_j^*, ct) \right] \right\}^{\frac{2}{n-1}} \tag{3.12}$$

and

$$u(x_j^*, t) = \left\{ \frac{(n+1)^2}{n(3n+1)} \sin^2 \left[\frac{n-1}{4cn} \sqrt{\frac{b_j^*}{k_j^*}} (x_j^*, ct) \right] \right\}^{\frac{2}{n-1}} \tag{3.13}$$

are readily obtained, where $x_1^* = x$, $x_2^* = x + y$, $x_3^* = x + y + z$.

However, for $b_j^* < 0$, we obtain the following solitary patterns solutions

$$u(x_j^*, t) = \left\{ \frac{(n+1)^2}{n(3n+1)} \cosh^2 \left[\frac{n-1}{4cn} \sqrt{\frac{b_j^*}{k_j^*}} (x_j^*, ct) \right] \right\}^{\frac{2}{n-1}},$$

$$a_j^*, k_j^* > 0, b_j^* < 0, c = \pm \left(\frac{a_j^* \pm \sqrt{(a_j^*)^2 + \frac{3(n+1)^4 b_j^*}{2k_j^* n^3 (3n+1)}}}{2} \right)^{\frac{1}{2}}, \tag{3.14}$$

and

$$u(x_j^*, t) = \left\{ \frac{(n+1)^2}{n(3n+1)} \sinh^2 \left[\frac{n-1}{4cn} \sqrt{\frac{b_j^*}{k_j^*}} (x_j^*, ct) \right] \right\}^{\frac{2}{n-1}},$$

$$a_j^*, k_j^* > 0, b_j^* < 0, c = \pm \left(\frac{a_j^* \pm \sqrt{(a_j^*)^2 + \frac{3(n+1)^4 b_j^*}{2k_j^* n^3 (3n+1)}}}{2} \right)^{\frac{1}{2}}. \tag{3.15}$$

Combining (3.11) with (2.1) and (2.2), the following compactons solutions

$$u(x_j^*, t) = \left\{ \pm \sqrt{\frac{1}{6}} \cos\left[\frac{1}{3} \sqrt{\frac{b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}},$$

and

$$u(x_j^*, t) = \left\{ \pm \sqrt{\frac{1}{6}} \sin\left[\frac{1}{3} \sqrt{\frac{b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}},$$

are readily obtained.

However, for $b_j^* < 0$, we obtain the following solitary patterns solutions

$$u(x_j^*, t) = \left\{ \pm \sqrt{\frac{1}{6}} \cosh\left[\frac{1}{3} \sqrt{\frac{-b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \; a_j^*, k_j^* > 0, \; b_j^* < 0, \quad (3.16)$$

and

$$u(x_j^*, t) = \left\{ \pm \sqrt{\frac{1}{6}} \sinh\left[\frac{1}{3} \sqrt{\frac{-b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \; a_j^*, k_j^* > 0, \; b_j^* < 0. \quad (3.17)$$

3.2. For negative exponents

We consider the variant $CH(-n, (-n, -m))$ of the Camassa-Holm equation

$$u_{tt} = (a_0 u + b_0 u^{-n})_{xx} + k_0 (u^{-n} + u^{-m})_{xxtt}, \quad (3.18)$$

$$u_{tt} = (a_0 u + b_0 u^{-n})_{xx} + (a_1 u + b_1 u^{-n})_{yy} + (k_0 (u^{-n} + u^{-m})_{xx} + k_1 (u^{-n} + u^{-m})_{yy})_{tt}, \quad (3.19)$$

$$u_{tt} = (a_0 u + b_0 u^{-n})_{xx} + (a_1 u + b_1 u^{-n})_{yy} + (a_2 u + b_2 u^{-n})_{zz} + (k_0 (u^{-n} + u^{-m})_{xx} + k_1 (u^{-n} + u^{-m})_{yy} + k_2 (u^{-n} + u^{-m})_{zz})_{tt}, \quad (3.20)$$

where $a_0, a_1, a_2, k_0, k_1, k_2 > 0$, $n \geq 2$, $n \neq m > 1$. Using the wave variable $\xi = x_j^* - ct$, carries (3.18), (3.19) and (3.20) into the ODE, respecyively,

$$c^2 u'' = (a_j^* u + b_j^* u^{-n})'' + c^2 k_j^* (u^{-n} + u^{-m})^{(4)}. \tag{3.21}$$

Integrating (3.21) twice, using the constants of integration to be zero we find

$$(a_j^* - c^2)u + b_j^* u^{-n} + c^2 k_j^* (u^{-n} + u^{-m})'' = 0. \tag{3.22}$$

Substituting (2.1) into (3.22) gives

$$\begin{aligned}(a_j^* - c^2)\lambda \cos^{\beta}(\mu\xi) + b_j^* \lambda^{-n} \cos^{-n\beta}(\mu\xi) + c^2 k_j^* (-n^2\mu^2\lambda^{-n}\beta^2 \cos^{-n\beta}(\mu\xi)\\ - n\mu^2\lambda^{-n}\beta(-n\beta - 1)\cos^{-n\beta-2}(\mu\xi) - m^2\mu^2\lambda^{-m}\beta^2 \cos^{-m\beta}(\mu\xi)\\ - m\mu^2\lambda^{-m}\beta(-m\beta - 1)\cos^{-m\beta-2}(\mu\xi)) = 0.\end{aligned} \tag{3.23}$$

Equation (3.23) is satisfied only if the following system of algebraic equations holds:

$$n\beta \neq -1,\ m\beta \neq -1, \beta = -m\beta - 2,\ m\beta = n\beta + 2,\ (a_j^* - c^2)\lambda - m\mu^2\lambda^{-m}\beta(-m\beta - 1) = 0,$$

$$b_j^*\lambda^{-n} - c^2 k_j^* n^2\mu^2\lambda^{-n}\beta^2 = 0, c^2 k_j^* n\mu^2\lambda^{-n}\beta(n\beta+1) - c^2 k_j^* m^2\mu^2\lambda^{-m}\beta^2 = 0, \tag{3.24}$$

or

$$n\beta \neq -1,\ m\beta = -1,\ m\beta = n\beta + 2,\ a_j^* - c^2 = 0,\ \lambda - m\mu^2\lambda^{-m}\beta(-m\beta - 1) = 0,$$

$$b_j^*\lambda^{-n} - c^2 k_j^* n^2\mu^2\lambda^{-n}\beta^2 = 0, c^2 k_j^* n\mu^2\lambda^{-n}\beta(n\beta+1) - c^2 k_j^* m^2\mu^2\lambda^{-m}\beta^2 = 0. \tag{3.25}$$

Solving the system (3.24) and (3.25) give, respectively,

$$\beta = \frac{-4}{n+1},\ n - 1 = 2m,\ \mu = \frac{n+1}{4cn}\sqrt{\frac{b_j^*}{k_j^*}},\ \lambda = \left(\frac{(n-1)^2}{n(3n-1)}\right)^{-\frac{2}{n+1}},$$

$$c = \pm\left(\frac{a_j^* \pm \sqrt{(a_j^*)^2 + \dfrac{3(n-1)^4 b_j^*}{2k_j^* n^3(3n-1)}}}{2}\right)^{\frac{1}{2}}, \; a_j^*, \; k_j^* > 0, \tag{3.26}$$

and

$$c = \pm\sqrt{a_j^*}, \; \beta = -\frac{1}{m}, \; n = 3m, \; \mu = \frac{1}{3}\sqrt{\frac{b_j^*}{a_j^* k_j^*}}, \; \lambda = \pm\left(\frac{1}{6}\right)^{-\frac{1}{2m}},$$

$$a_j^*, \; k_j^* > 0. \tag{3.27}$$

The results (3.26) and (3.27) can be easily obtained, if we also use the sine method (2.2). Combining (3.26) with (2.1) and (2.2), the following compactons solutions

$$u(x_j^*, t) = \left\{\frac{n(3n-1)}{(n-1)^2}\sec^2\left[\frac{n+1}{4cn}\sqrt{\frac{b_j^*}{k_j^*}}(x_j^* - ct)\right]\right\}^{\frac{2}{n+1}}, \tag{3.28}$$

and

$$u(x_j^*, t) = \left\{\frac{n(3n-1)}{(n-1)^2}\csc^2\left[\frac{n+1}{4cn}\sqrt{\frac{b_j^*}{k_j^*}}(x_j^* - ct)\right]\right\}^{\frac{2}{n+1}}. \tag{3.29}$$

However, for $b_j^* < 0$, we obtain the following solitary patterns solutions

$$u(x_j^*, t) = \left\{\frac{n(3n-1)}{(n-1)^2}\operatorname{sech}^2\left[\frac{n+1}{4cn}\sqrt{\frac{b_j^*}{k_j^*}}(x_j^* - ct)\right]\right\}^{\frac{2}{n+1}},$$

$$a_j^*, \; k_j^* > 0, \; b_j^* < 0, \; c = \pm\left(\frac{a_j^* \pm \sqrt{(a_j^*)^2 + \dfrac{3(n-1)^4 b_j^*}{2k_j^* n^3(3n-1)}}}{2}\right)^{\frac{1}{2}}, \tag{3.30}$$

and

$$u(x_j^*, t) = \left\{ \frac{n(3n-1)}{(n-1)^2} \operatorname{csch}^2 \left[\frac{n+1}{4cn} \sqrt{\frac{b_j^*}{k_j^*}} (x_j^* - ct) \right] \right\}^{\frac{2}{n+1}},$$

$$a_j^*, k_j^* > 0, b_j^* < 0, c = \pm \left(\frac{a_j^* \pm \sqrt{(a_j^*)^2 + \frac{3(n-1)^4 b_j^*}{2k_j^* n^3 (3n-1)}}}{2} \right)^{\frac{1}{2}}. \tag{3.31}$$

Combining (3.27) with (2.1) and (2.2), the following compactons solutions

$$u(x_j^*, t) = \left\{ \sqrt{6} \sec \left[\frac{1}{3} \sqrt{\frac{b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \tag{3.32}$$

and

$$u(x_j^*, t) = \left\{ \sqrt{6} \csc \left[\frac{1}{3} \sqrt{\frac{b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \tag{3.33}$$

are readily obtained.

However, for $b_j^* < 0$, we obtain the following solitary patterns solutions

$$u(x_j^*, t) = \left\{ \sqrt{6} \operatorname{sech} \left[\frac{1}{3} \sqrt{\frac{-b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \; a_j^*, k_j^* > 0, b_j^* < 0, \tag{3.34}$$

and

$$u(x_j^*, t) = \left\{ \sqrt{6} \operatorname{csch} \left[\frac{1}{3} \sqrt{\frac{-b_j^*}{a_j^* k_j^*}} (x_j^* \pm \sqrt{a_j^*} t) \right] \right\}^{\frac{1}{m}}, \; a_j^*, k_j^* > 0, b_j^* < 0. \tag{3.35}$$

4. Discussion

The solitary wave and compactons solutions for a class of nonlinear fourth order variant of a generalized Camassa-Holm equation is obtained analytically by using the sine-cosine method. The obtained results in this work clearly demonstrate the effect of the purely nonlinear dispersion and the qualitative change made in the genuinely nonlinear phenomenon. This approach may be applied to seek travelling wave solutions for other types of nonlinear dispersion partial differential equations which satisfy certain restrictions.

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